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Minimal representations of supersymmetry and 1D N-extended σ-models

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Minimal representations of supersymmetry and $1D$ N-extended σ-models *

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We discuss the minimal representations of the $1D$ N-Extended Supersymmetry algebra (the Z_2-graded symmetry algebra of the Supersymmetric Quantum Mechanics) linearly realized on a finite number of fields depending on a real parameter t, the time. Their knowledge allows to construct one-dimensional sigma-models with extended off-shell supersymmetries without using superfields.

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I. INTRODUCTION

The superalgebra of the Supersymmetric Quantum Mechanics ($1D$ N-Extended Supersymmetry Algebra) [1] is a $\mathbf{Z}_2$-graded algebra expressed by N odd generators Q_i ($i = 1, \ldots, N$) and a single even generator H (the hamiltonian). It is defined by the (anti)-commutation relations

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij}H, \\ [Q_i, H] &= 0. \end{aligned} \tag{1}$$

The structure of its minimal linear representations realized on a finite number of fields depending on a single real parameter (t, the time) has been substantially elucidated in recent years. Several results have been obtained [2–7]. They are based on the Atiyah-Bott-Shapiro [8] classification of the irreducible Clifford algebras. In this paper we discuss several results on the classification of the (1) minimal representations and their use in constructing one-dimensional sigma-models with extended number of off-shell supersymmetries.

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The problem we addressing can be stated as follows: to construct and classify, for any given integer N, the linear representations of (1) acting on a finite, minimal, number of fields, even and odd (bosonic and fermionic), depending on t. The generator H has to be represented by a time-derivative, while the Q_i's generators must be realized by finite-dimensional matrices whose entries are either c-numbers or time-derivatives up to a certain power. The representation space we are considering is infinite-dimensional, being given by the set of fundamental fields and their time-derivatives of any order. In the physical literature these representations are called "finite" since they are obtained by a finite number of fundamental fields (the situation parallels here the representation theory of chiral algebras [9], given by the generating set of primary fields and their descendants; the time-derivatives of the fundamental fields play, for (1) representations, the role of the descendants in chiral algebra representations). For the same reason, the notion of "minimal representations" is expressed, in the physical literature, as "irreducible representations".

The program of classifying the (1) minimal representations starts with [2], with the recognition that formulating an eigenvalue problem for the hamiltonian H (for an eigenvalue different from zero) reduces the Q_i's anticommutators to, up to normalization, the basic relation for Euclidean Clifford algebra generators. The [2] main result can be stated as follows. The minimal representations of (1), for a given N, are obtained by applying a dressing transformation to a fundamental representation (nowadays called in the literature the "root multiplet"), with equal number of bosonic and fermionic fields. The root multiplet is specified by an associated Euclidean Clifford algebra. As a main corollary, the total number n of bosonic fundamental fields entering a minimal representation equals the total number of fermionic fundamental fields and is expressed, for any given N, by the following relation [2]

$$\begin{aligned} N &= 8l+m, \\ n &= 2^{4l}G(m), \end{aligned} \tag{2}$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$.

$G(m)$ appearing in (2) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

(3)

Note the *mod* 8 Bott's periodicity.

An integral $\mathbf{Z}$-grading, compatible with the $\mathbf{Z}_2$-grading of the superalgebra, can be assigned to the fundamental fields and their time-derivatives. In the physical literature, the grading is referred as "mass-dimension". The integral grading will be denoted by z. For convenience, the mass-dimension d will be expressed as $d = \frac{z}{2}$. The hamiltonian H has mass-dimension $d = 1$ (its fermionic roots, the Q_i's operators, have mass-dimension $d = \frac{1}{2}$). Bosonic (fermionic) fields have integer (respectively, half-integer) mass-dimension. Each linear representation admitting a finite number of fundamental fields is characterized by its "fields content", i.e. the set of integers $(n_1, n_2, \ldots, n_l)$ specifying the number n_i of fundamental fields of dimension d_i ($d_i = d_1 + \frac{i-1}{2}$, with d_1 an arbitrary constant) entering the representation. Physically, the n_l fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is defined to be the length of the representation (a root representation has length $l = 2$). Either $n_1, n_3, \ldots$ correspond to the bosonic fields (therefore $n_2, n_4, \ldots$ specify the fermionic fields) or viceversa. In both cases the equality $n_1 + n_3 + \ldots = n_2 + n_4 + \ldots = n$ is guaranteed.

The representation theory does not discriminate the overall bosonic or fermionic nature of the representation.

According to [2], if $(n_1, n_2, \ldots, n_l)$ specifies the fields content of an irreducible representation, $(n_l, n_{l-1}, \ldots, n_1)$ specifies the fields content of a dual irreducible representation. Representations such that $n_1 = n_l, n_2 = n_{l-1}, \ldots$ are called "self-dual representations". In [3] it was shown how to extract from the associated Clifford algebras the admissible fields content of the (1) linear finite irreducible representations. We discuss these results in the next Section.

II. SUPERSYMMETRIC QUANTUM MECHANICS AND CLIFFORD ALGEBRAS

In this Section we give a more detailed account of the connection between representations of the Supersymmetric Quantum Mechanics and Clifford algebras.

According to [2] the length-2 minimal representations of the (1) supersymmetry algebra are uniquely determined by a representation of an associated Clifford algebra. The connection goes as follows. The supersymmetry generators acting on a length-2 irreducible multiplet can be expressed as

$$Q_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i \cdot H & 0 \end{pmatrix} \tag{4}$$

where σ_i and $\widetilde{\sigma}_i$ are matrices entering a Weyl type (i.e. block antidiagonal) irreducible representation of the Clifford algebra relation

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i & 0 \end{pmatrix} \quad , \quad \{\Gamma_i, \Gamma_j\} = 2\eta_{ij} \tag{5}$$

The Q_i's in (4) are supermatrices with vanishing bosonic and non-vanishing fermionic blocks, acting on a multiplet m (thought of as a column vector) which can be either bosonic or fermionic (we conventionally consider a length-2 irreducible multiplet as bosonic if its upper half part of component fields is bosonic and its lower half is fermionic; it is fermionic in the converse case). The connection between Clifford algebra irreps of the Weyl type and minimal representations with minimal length of the N-extended one-dimensional supersymmetry is such that D, the dimensionality of the (Euclidean, in the present case) space-time of the Clifford algebra (5) coincides with the number N of the extended supersymmetries, according to

♯ of space-time dim. (Weyl-Clifford)	⇔	♯ of extended su.sies (in 1-dim.)
D	$=$	N

(6)

The matrix size of the associated Clifford algebra (equal to $2n$, with n given in (2)) corresponds to the number of (bosonic plus fermionic) fields entering the one-dimensional N-extended supersymmetry irrep.

The classification of Weyl-type Clifford irreps, furnished in [2], can be easily recovered from the well-known classification of Clifford irreps, given in [8] (see also [10] and [11]).

The (4) Q_i's matrices realizing the N-extended supersymmetry algebra (1) on length-2 minimal representations have entries which are either c-numbers or are proportional to the Hamiltonian H. Minimal representations of higher length ($l \geq 3$) are systematically produced [2] through repeated applications of the dressing transformations

$$Q_i \mapsto \widehat{Q}_i^{(k)} = S^{(k)} Q_i {S^{(k)}}^{-1} \tag{7}$$

realized by diagonal matrices $S^{(k)}$'s $(k = 1, \ldots, 2n)$ with entries ${s^{(k)}}_{ij}$ given by

$$ {s^{(k)}}_{ij} \;=\; \delta_{ij}(1 - \delta_{jk} + \delta_{jk}H) \tag{8} $$

Some remarks are in order [2]

i) the dressed supersymmetry operators ${Q_i}'$ (for a given set of dressing transformations) have entries which are integral powers of H. A subclass of the ${Q_i}'$ s dressed operators is given by the local dressed operators, whose entries are *non-negative* integral powers of H (their entries have no $\frac{1}{H}$ poles). A local representation (minimal representations fall into this class) of an extended supersymmetry is realized by local dressed operators. The number of the extension, given by N' $(N' \leq N)$, corresponds to the number of local dressed operators.

ii) The local dressed representation is not necessarily a minimal representation. Since the total number of fields (n bosons and n fermions) is unchanged under dressing, the local dressed representation is a minimal representation iff n and N' satisfy the (2) requirement (with N' in place of N).

iii) The dressing changes the dimension of the fields of the original multiplet m. Under the $S^{(k)}$ dressing transformation (7), $m \mapsto S^{(k)}m$, all fields entering m are unchanged apart from the k-th one (denoted, e.g., as φ_k and mapped to $\dot{\varphi}_k$). Its dimension is changed from $[k] \mapsto [k]+1$. This is why the dressing changes the length of a minimal representation. As an example, if the original length-2 multiplet m is a bosonic multiplet with d 0 mass-dimension bosonic fields and d $\frac{1}{2}$ mass-dimension fermionic fields (in the following such a multiplet will be denoted as $(x_i; \psi_j) \equiv (d, d)_{s=0}$, for $i, j = 1, \ldots, d$), then $S^{(k)}m$, for $k \leq d$, corresponds to a length-3 multiplet with $d-1$ bosonic fields of 0 mass-dimension, d fermionic fields of $\frac{1}{2}$ mass-dimension and a single bosonic field of mass-dimension 1 (in the following we employ the notation $(d-1, d, 1)_{s=0}$ for such a multiplet of fields).

When looking purely at the representation properties of a given multiplet the assignment of an overall mass-dimension s is arbitrary, since the supersymmetry transformations of the fields are not affected by s. Introducing an overall mass-dimension is useful for tensoring multiplets and becomes essential for physical applications, e.g. in the construction of supersymmetric invariant terms entering an action.

In the above multiplet l denotes its length, d_l the number of auxiliary fields of highest mass-dimension transforming as time-derivatives. The total number of odd-indexed equal the total number of even-indexed fields, i.e. $d_1 + d_3 + \ldots = d_2 + d_4 + \ldots = d$. The multiplet is bosonic if the odd-indexed fields are bosonic and the even-indexed fields are fermionic (the multiplet is fermionic in the converse case). For a bosonic multiplet the auxiliary fields are bosonic (fermionic) if the length l is an odd (even) number.

Just like the overall mass-dimension assignment, the assignment of a bosonic (fermionic) character to a multiplet is arbitrary since the mutual transformation properties of the fields inside a multiplet are not affected by its statistics. Therefore, multiplets always appear in dually related pairs so that to any bosonic multiplet there exists its fermionic counterpart with the same transformation properties.

Throughout this paper we assign integer valued mass-dimension to bosonic multiplets and half-integer valued mass-dimension to fermionic multiplets.

As pointed out before, the most general $(d_1, d_2, \ldots, d_l)$ multiplet is recovered as a dressing of its corresponding N-extended length-2 (d, d) multiplet. In [2] it was shown that all dressed supersymmetry operators producing any length-3 multiplet (of the form $(d-p, d, p)$ for $p = 1, \ldots, d-1$) are of local type. Therefore, for length-3 multiplets, we have $N' = N$. This implies, in particular, that the $(d-p, d, p)$ multiplets are non-equivalent irreps of the N-extended one-dimensional supersymmetry. As concerns length $l \geq 4$ multiplets, the general problem of finding minimal representations was not addressed in [2]. It was shown, as a specific example, that the dressing of the length-2 $(4, 4)$ irrep of $N = 4$, realized through the series of mappings $(4, 4) \mapsto (1, 4, 3) \mapsto (1, 3, 3, 1)$, produces at the end a length-4 multiplet $(1, 3, 3, 1)$ carrying only three local supersymmetries ($N' = 3$). Since the relation (2) is satisfied when setting the number of extended supersymmetries acting on a multiplet equal to 3 and the total number of bosonic (fermionic) fields entering a minimal representation equal to 4, as a consequence, the $(1, 3, 3, 1)$ multiplet corresponds to a minimal representation of the $N = 3$ extended supersymmetry.

Based on an algorithmic construction of representatives of Clifford irreps, an iterative method to compute the admissible field contents of the minimal representations for arbitrary N values of the extended supersymmetry was presented in [3].

III. SUPERSYMMETRY GRAPHS AND THEIR CONNECTIVITY

In this Section we describe, largely based on [12], the graphical interpretation of the minimal supersymmetry representations and discuss, based on [6] and [7], their connectivity properties.

An association can be made between N-colored oriented graphs and the linear supersymmetry transformations. The identification goes as follows. The fundamental fields (bosonic and fermionic) entering a representation are expressed as vertices. They can be accommodated into an $X-Y$ plane. The Y coordinate can be chosen to correspond to the mass-dimension d of the fields. Conventionally, the lowest dimensional fields can be associated to vertices lying on the X axis. The higher dimensional fields have positive, integer or half-integer values of Y. A colored edge links two vertices which are connected by a supersymmetry transformation. Each one of the N Q_i supersymmetry generators is associated to a given color. The edges are oriented. The orientation reflects the sign (positive or negative) of the corresponding supersymmetry transformation connecting the two vertices. Instead of using arrows, alternatively, solid or dashed lines can be associated, respectively, to positive or negative signs. No colored line is drawn for supersymmetry transformations connecting a field with the time-derivative of a lower dimensional field. This is in particular true for the auxiliary fields (the fields of highest dimension in the representation) which are necessarily mapped, under supersymmetry transformations, in the time-derivative of lower-dimensional fields.

Each irreducible supersymmetry transformation can be presented (the identification is not unique) through an oriented N-colored graph with $2n$ vertices (see (2)). The graph is such that precisely N edges, one for each color, are linked to any given vertex which represents either a 0-mass dimension or a $\frac{1}{2}$-mass dimension field.

Despite the fact that the presentation of the graph is not unique, certain of its features only depend on the class of the supersymmetry transformations. We introduce now, following [6], the invariant characterization. An unoriented "color-blind" graph can be associated to the initial graph by disregarding the orientation of the edges and their colors (all edges are painted in black). For simplicity, we discuss here the invariant characterization of the graphs associated to the length $l = 3$ irreducible representation that will be discussed in the following (the generalization of the invariant characterization to graphs of arbitrary length is straightforward, see [6]). They admit fields content $(k, n, n-k)$. The corresponding fields are denotes as x_p (for 0-mass dimension), ψ_q (for $\frac{1}{2}$ mass-dimension) and g_r (the 1 mass-dimension auxiliary fields), where $p =, 1, \ldots, k$, $q = 1, \ldots, n$ and $r = 1, \ldots, n-k$.

The connectivity of the associated length $l = 3$ color-blind graph can be expressed through the connectivity symbol ψ_g, expressed as

$$\psi_g \;=\; (m_1)_{s_1} + (m_2)_{s_2} + \ldots + (m_Z)_{s_Z}. \tag{9}$$

The ψ_g symbol encodes the information on the partition of the n $\frac{1}{2}$-mass dimension fields (vertices) into the sets of m_z vertices ($z = 1, \ldots, Z$) with s_z edges connecting them to the $n-k$ 1-mass dimension auxiliary fields. We have

$$m_1 + m_2 + \ldots + m_Z \;=\; n, \tag{10}$$

while $s_z \neq s_{z'}$ for $z \neq z'$.

The connectivity symbol is an invariant characterization of the class of the irreducible supersymmetry transformations.

The connectivity symbol ψ_g can be used to induce a map $\widetilde{\psi_g}$ from the set of graphs Gr into the set of integers $\mathbf{Z}$ ($\widetilde{\psi}_g : Gr \rightarrow \mathbf{Z}$) s.t. $W \in \mathbf{Z}$ is given by

$$W \;=\; \prod_{z=1}^{Z} (p_{2z-1}^{m_z})(p_{2z}^{s_z}), \tag{11}$$

where the p_w's, $w = 1, 2, 3, \ldots$, denote the ordered set of prime integers $(2, 3, 5, \ldots)$. With the above definition two inequivalent connectivities induce two distinct integers W, W' ($W' \neq W$).

IV. $1D$ σ-MODELS WITH OFF-SHELL SUPERSYMMETRIES

The results discussed so far concerning the construction and classification of the minimal representations of the (1) supersymmetry algebra, allow us to introduce a different and more comprehensive approach than the one based on superfields for the construction of one-dimensional sigma-models with N extended supersymmetries. Following [13] we discuss it in connection with σ-models with $N = 5, 6, 7, 8$ off-shell supersymmetries. In these cases the minimal representations involve a total number of 8 bosonic and 8 fermionic fields. According to [2] and [3] the length-3 representations are labeled by $(k, 8, 8-k)$, with $k = 1, 2, \ldots, 7$. According to [6] and [7], for $N = 5, 6$ and $k = 2, 3, 4, 5, 6$, supersymmetry transformations are found with inequivalent connectivity. Using an $N = 8$ superfield approach, a partial list of results was produced in [14] for $N = 8$ $1D$ sigma-models. The alternative construction of [13] allows to solve some unanswered questions, e.g., whether inequivalent connectivities induce different off-shell actions. The scheme of [13] heavily relies on a computational package for Maple 11 which allows to deal with anticommuting fields.

We illustrate here the basic strategy. At first we have to construct the most general homogeneous term $\mathcal{T}_d$ of mass-dimension d, constructed in terms of the bosonic 0-dimensional fields x_i, the $\frac{1}{2}$-dimensional fermionic fields ψ_j, the 1-dimensional auxiliary fields g_l and their time-derivatives (a time-derivative counts as 1 in mass-dimension). No matter which is the value of k, the following number of independent functions is encountered at each level d.

$$\begin{aligned}
\mathcal{T}_0 &: 1 \quad function, \\
\mathcal{T}_{\frac{1}{2}} &: 8 \quad functions, \\
\mathcal{T}_1 &: 36 \quad function, \\
\mathcal{T}_{\frac{3}{2}} &: 128 \quad functions, \\
\mathcal{T}_2 &: 402 \quad functions, \\
\mathcal{T}_{\frac{5}{2}} &: 1152 \quad functions.
\end{aligned} \tag{12}$$

We recall that, for $d = 2$, we have the correct mass-dimension for a sigma-model kinetic term.

A manifestly $\overline{N}$-extended supersymmetric lagrangian $\mathcal{L}_{\overline{N}}$ is produced through the position

$$\mathcal{L}_{\overline{N}} = Q_1 \cdots Q_{\overline{N}} F_{\overline{N}}, \tag{13}$$

with, in mass-dimension,

$$\begin{aligned}
[\mathcal{L}_{\overline{N}}] &= 2, \\
[F_{\overline{N}}] &= 2 - \frac{\overline{N}}{2}.
\end{aligned} \tag{14}$$

One can define the action $\mathcal{S} = \int dt \mathcal{L}_{\overline{N}}$ provided that $\mathcal{L}_{\overline{N}}$ is not a total derivative. The action $\mathcal{S}$ is N-extended supersymmetric if, moreover, $N - \overline{N}$ constraints are satisfied, for $j = \overline{N} + 1, \cdots, N$, such that

$$Q_j \mathcal{L}_{\overline{N}} = \partial_t R_{j,\overline{N}}, \tag{15}$$

where, in mass-dimension, we have

$$[R_{j,\overline{N}}] = \frac{3}{2}. \tag{16}$$

Imposing a supersymmetry constraint for each extra supersymmetry operator produces a system of 1152 constraining equations to be solved in terms of 128 functions (the coefficients entering $R_{j,\overline{N}}$). Needless to say, the great majority of these 1152 constraints are trivially satisfied, while many other constraints are redundant (the same constraint is repeated over and over). It should be noticed that, according to our notion of manifest supersymmetry, $N = 4$ off-shell supersymmetries can always assumed to be manifest. On the other hand, starting from $N \geq 5$, there exists extra supersymmetries which have to be imposed constraining the functions entering $F_{\overline{N}}$ (the total numbers are $1, 8, 36, 128, 402$ for $\overline{N} = 0, 1, 2, 3, 4$, respectively). In [13] we discussed in detail the most general off-shell action for $N = 5, 6, 7, 8$ and $k = 2$ (namely the $(2, 8, 6)$ multiplets, with their associated inequivalent connectivities). The detailed discussion of the results will be given in that reference. It should be mentioned that the scheme here proposed is a generalization of the one used in [3], where the most general $N = 8$ off-shell action for the $(1, 8, 7)$ multiplet was obtained (the problem there was somehow simplified, instead of producing a systematic analysis of the constraining equations, one could rely on the special symmetry properties induced by the octonionic structure constants). It is worth noticing that the systematic approach here outlined could in principle lead us to attack the problem of constructing off-shell actions beyond the $N = 8$ barrier. In particular we have in mind a special case of the $(9, 16, 7)$ multiplet with $N = 9$ off-shell supersymmetries. It is known to correspond (see [7]) to a $(D = 1)$ dimensional reduction of the $N = 4$ Super-Yang-Mills theory.

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