CBPF-NF-015/08

Triviality - quantum decoherence of Fermionic quantum chromodynamics $SU(N_c)$ in the presence of an external strong $U(\infty)$ flavored constant noise field

Luiz C.L. Botelho
Departamento de Matemática Aplicada,
Instituto de Matemática, Universidade Federal Fluminense,
Rua Mario Santos Braga
24220-140, Niterói, Rio de Janeiro, Brazil
PACS number(s): 11.15.-q, 12.38.Aw, 12.40.Yx

Abstract: We analyze the triviality-quantum decoherence of Euclidean quantum chromodynamics in the gauge invariant quark current sector in the presence of an external $U(\infty)$-flavor constant charged white noise reservoir.

Key-words: Triviality in QFT; Quantum decoherence; Nonperturbative QCD; Path integral.

1 Introduction

Recently, we have proposed a bosonic loop space formalism for understanding the important problem of triviality in interacting Gauge Field theories ([1], [2]). The basic idea used in our work above mentioned in order to analyze such kind of quantum triviality phenomena was the systematic use of the framework of the loop space to rewrite particle-field path integrals in terms of its ensembre of quantum trajectories and the introduction of a noisely electromagnetic field as an external quantized reservoir.

The purpose of this letter is to point out quantum field triviality phenomena in the context of our previous loop space formalism for the case of Fermionic Quantum Chromodynamics with finite number of colors but in presence of an external non-abelian translation independent $U(\infty)$-flavor charged white noise simulating a quantum field reservoir ([1]).

In order to show exactly this triviality result for $Q.C.D(SU(N_c))$ in such a context of an external non-abelian reservoir, we use of Migdal-Makeenko loop space expression for the spin quark generating functional of abelian vectorial quarks currents ([3]) – associated to the physical abelian vectorial mesons, added with the explicitly evaluation of $U(M)$-flavor Wilson Loops at the t'Hooft $M \to \infty$ limit for translation invariant noise-flavor field configurations.

We finally arrive at our main result that the triviality of Quantum Chromodynamics at such a kind of flavor reservoir, is linked to the problem of quantum decoherence in Quantum Physics ([1]). In appendix A, we present an aplication of our study to the Physical Problem of Confining in Yang-Mills Theory. In appendix B, we present the detailed analysis of the problem of large N in Statistics, which mathematical ideas have underlyning our Path-Integral Analysis in the bulk of this letter.

2 The Triviality - Quantum Decoherence Analysis for Quantum Chromodynamics

In order to show such a triviality - quantum decoherence on Fermionic $Q.C.D(SU(N_c))$ with finite number of colors in the presence of $U(\infty)$ flavored random reservoirs, let us consider the physical Euclidean generating functional of the Abelian quarks currents in the presence of an external translation invariant white-noise $U(M)$ non-abelian field $B_\mu^{(M)}$, considered here as a kind of "dissipative" non-abelian reservoir structure and corresponding to the interaction quarks flavor charges with a $U(M)$ vacuum-reservoir structure, namely

$$Z[J_\mu(x), B_\mu^{(M)}] = \left\langle \det{}_F \begin{bmatrix} 0 & \not{D}(A_\mu, B_\mu^{(M)}, J_\mu) \\ \not{D}^*(A_\mu, B_\mu^{(M)}, J_\mu) & 0 \end{bmatrix} \right\rangle_{A_\mu} \tag{1}$$

Here the Euclidean Dirac operator is explicitly given by

$$\not{D}(A_\mu, B_\mu^{(M)}, J_\mu) = i\gamma_\mu(\partial_\mu + g^{(M)} B_\mu^{(M)} + eA_\mu + J_\mu) \tag{2}$$

with $eA_\mu(x)$ denoting the $SU(N_c)$ Yang-Mills non-Abelian quantum field (translation dependent) configurations averaged in eq.(1) by means of the usual Yang-Mills path integral denoted by $\langle \quad \rangle_{A_\mu}$, $J_\mu(x)$ is the auxiliary source field associated to the abelian quark currents and $g^{(M)} B_\mu^{(M)}$ is a random translation invariant external $U(M)$ flavor Yang-Mills field with a constant field strength

$$F_{\mu\nu}(B) = (ig_M)[B_\mu^{(M)}, B_\nu^{(M)}]. \tag{3}$$

Here $E_F^{(M)}$ denotes the stochastic average on the ensemble of the external random $U(M)$ non-abelian strenght fields defined by the $U(M)$-invariant path-integral ([4])

$$\begin{aligned} E_F^{(M)}\{\mathcal{O}(B_\mu)\} \equiv \frac{1}{E_F^{(M)}\{1\}} & \bigg(\int \Big(\prod_{\mu=1}^{D} \prod_{a=1}^{M^2} dB_\mu^a \Big) \\ & \times \exp\left\{ -\frac{1}{2} \left[(ig_M)^2 \mathrm{Tr}_{U(M)}([B_\mu^{(M)}, B_\nu^{(M)}]^2) \right] \right\} \\ & \times \mathcal{O}(B_\mu^{(M)}) \bigg) \end{aligned} \tag{4}$$

with $\mathcal{O}(B_\mu^{(M)})$ denoting an $U(M)$-flavor invariant observable on the presence of an external translation invariant random $U(M)$-valued non-abelian reservoir field $B_\mu^{(M)}$.

In the fermionic loop space framework ([1], [2], [3]), we can express the quark functional determinant, eq.(1) – which has been obtained as an effective generating functional for the color N_c-singlet quark current after integrating out the Euclidean quark action – as a purely functional on the bosonic bordered loop space composed of all trajectories $C_{xx} = \{X_\mu(\sigma), X_\mu(0) = X_\mu(T) = x; 0 \le \sigma \le T\}$, namely

$$\begin{aligned} Z[J_\mu(x), B_\mu^{(M)}] = \Big\langle \exp\Big\{ - N_c \mathrm{spur}\Big[\sum_{C_{xx}} \mathbb{P}_{\mathrm{Dirac}} \Big[\exp\Big(\oint_{C_{xx}} d\sigma \frac{i}{2} [\gamma^\mu, \gamma^\nu](\sigma) \\ \times \frac{\delta}{\delta\sigma_{\mu\nu}(X(\sigma))} \Big] \times \mathrm{Tr}_{U(M)}(\Phi[C_{xx}, B_\mu^{(M)}]) \Big\} \\ \times \Phi[C_{xx}, J_\mu] \times \mathrm{Tr}_{SU(N_c)}(W[C_{xx}, A_\mu]) \Big) \Big] \Big\} \end{aligned} \tag{5}$$

where $\Phi[C_{xx}, B_\mu^{(M)}]$ is the usual Wilson-Mandelstam path-ordered loop variable defined by the translation invariant random external (reservoir) $U(M)$ field $B_\mu^{(M)}$, and $W[C_{xx}, A_\mu]$ is the same loop space object for the dynamical quantum color gauge field $SU(N_c)$.

Note the appearance of the Migdal-Makeenko area – loop derivative operator with the Dirac index path ordenation in order to take into account explicitly the relevant spin-orbit interaction of the quarks Dirac spin with the set of interacting vectorial fields $\{A_\mu(x), B_\mu, J_\mu(x)\}$ in the theory described by eq.(1) ([3]) (the well-know bordered loops).

The sum over the closed bosonic loops C_{xx}, with end-point x is given by the proper-time bosonic path integral below ([1],[2])

$$\sum_{C_{xx}} = \int_0^\infty \frac{dT}{T} \int d^D x \int_{X(0)=x=X(T)} D^F[X(\sigma)] \times \exp\left\{-\frac{1}{2}\int_0^T \dot{X}^2(\sigma)d\sigma\right\}. \tag{6}$$

Following the idea of our previous work on Triviality-Quantum Decoherence of Gauge theories [1], we need to show in eq.(5) that at the t'Hooft topological limit of $M \to \infty$ in the ensemble of external white-noise reservoir fields $B_\mu^{(M)}$ as implemented in ref. [1], one obtains for the Wilson Loop $E_F(\Phi[C_{xx}, B_\mu^{(M)}])$ an area-power behavior on the (minimal) area $S[C_{xx}]$ bounded by the large area loops C_{xx} inside the loop space functional on eq.(5), after considering the average of the infinite-flavor limit on the external translation independent white-noise B_μ field eq.(3)–eq.(4).

In the context of a cummulant expansion for the loop space integrand in eq.(5) defined by the $U(M)$ path integral eq.(4), one should firstly evaluate the following Wilson Loop path integral (loop normalized to unity) on the $U(M)$-noise reservoir field $B_\mu^{(M)}$:

$$\begin{aligned} & E_F^{(M)}\{\mathrm{Tr}_{U(M)}(\phi[C_{xx}, B_\mu^{(M)}])\} \\ & = \frac{1}{E_F^{(M)}\{1\}} \int_{-\infty}^{+\infty} \left(\prod_{a=1}^{M^2}\prod_{\mu=1}^{D} dB_\mu^{a,(M)}\right) \\ & \times \exp\left\{+\frac{1}{2}(g_M)^2 \mathrm{Tr}_{U(M)}([B_\mu^{(M)}, B_\nu^{(M)}]^2)\right\} \\ & \times \frac{1}{M}\mathrm{Tr}_{U(M)}\mathbb{P}\left\{e^{ig_M \oint_{C_{xx}} B_\mu^{(M)} dX_\mu}\right\}. \end{aligned} \tag{7}$$

By using the non-abelian Stokes theorem for constant gauge fields, one obtains the following result for large M ([4]):

$$\frac{1}{M}\left(\mathrm{Tr}_{SU(M)}\mathbb{P}\left\{e^{ig_M \oint_{C_{xx}} B_\mu^{(M)} dx_\mu}\right\}\right) = \frac{1}{M}\left(\mathrm{Tr}_{SU(M)}\mathbb{P}\left\{e^{ig_M \int_{S[C_{xx}]} F_{12}(B^{(M)}) S^{12}}\right\}\right) \tag{8-a}$$

or equivalently:

$$\frac{1}{M}\mathrm{Tr}_{SU(M)}\left(\mathbb{P}e^{-(g_M)^2[B_1,B_2]S[C_{xx}]}\right) = \exp\left\{+\frac{(g_M^2 S[C_{xx}])^2}{2M}(\mathrm{Tr}[B_1^{(M)}, B_2^{(M)}])^2\right\} + O(\frac{1}{M}) \tag{8-b}$$

where we have choosen the large loop C_{xx} to be contained in the plane $\mu = 1$, $\nu = 2$ without loss of generality.

A simple field re-scaling on the path-integral eq.(7) as written below, after inserting the $M \to \infty$ leading exact result of the Wilson Loop noise factor eq.(8) on the cited equation (7):

$$B^a_{\mu=1} \to \tilde{B}^a_{\mu=1}\left[g_M^2 + \frac{(g_M^2 S[C_{xx}])^2}{M}\right]^{-\frac{1}{4}} \tag{9}$$

$$B^a_{\mu=2} \to \tilde{B}^a_{\mu=2}\left[g_M^2 + \frac{(g_M^2 S[C_{xx}])^2}{M}\right]^{-\frac{1}{4}} \tag{10}$$

$$B^a_{\mu\neq\{1,2\}} \to \tilde{B}^a_{\mu\neq\{1,2\}}[g_M^2]^{-\frac{1}{4}} \tag{11}$$

leads us to the exactly result at the t'Hooft limit of $U(\infty)$ flavor charge

$$\begin{aligned}\lim_{M\to\infty}(E_F^{(M)}\{\mathrm{Tr}_{U(M)}(\Phi[C_{xx}, B^{(M)}_\mu])\}) &= \lim_{M\to\infty}\left\{\frac{\left[g_M^2\left(1 + \frac{g_M^2 S[C_{xx}]^2}{M}\right)\right]^{-\frac{1}{2}M^2}}{[g_M^2]^{-(\frac{(M^2 D)}{4})}}[g_M^2]^{-\frac{M^2(D-2)}{4}}\right\} \\ &= \exp\left\{-\frac{1}{4}(g_\infty)^2 S^2[C_{xx}]\right\} + O(\frac{1}{M})\end{aligned} \tag{12}$$

where $g_\infty^2 = \lim_{n\to\infty}((g_M)^2 M) < \infty$ denotes the $U(\infty)$-flavor reservoir t'Hooft coupling constant. Note that we have used the leading $M \to \infty$ limit on the weight on the numerator of the reservoir field path integral eq.(7). For instance (here $B_\mu \equiv B^a_\mu \lambda_a$ with $[\lambda_a, \lambda_b] = f_{abc}\lambda_c$)

$$\begin{aligned}&\lim_{n\to\infty}\left\{\exp\left[\left(\frac{1}{2}(g_M)^2 + \frac{(g_M)^4(S[C_{xx}])^2}{2M}\right)\times\left(\tilde{B}^a_1\tilde{B}^b_2\tilde{B}^{a'}_1\tilde{B}^{b'}_2 f_{abc}f^{ca'b'}\right)\right]\right\} \\ &\sim \exp\left\{\left[\left(\frac{1}{2}(g_M)^2\right)\times\tilde{B}^a_1\tilde{B}^b_2\tilde{B}^{a'}_1\tilde{B}^{b'}_2 f_{abc}f^{ca'b'}\right]\right\} \\ &+ O\left(\frac{1}{M}\right)\end{aligned} \tag{13}$$

which produces as the only non-trivial result at $M \to +\infty$ in the average eq.(7), that one arising from the ratio of the Jacobians of the measure change associated to re-scalings eq.(9)–eq.(11) on the path integral numerator eq.(7) and the normalization path-integral denominator respectively.

As a result, we get an exponential behavior for our noise $U(\infty)$-averaged Wilson Loop with an power square area argument.

Finally, we can see that the loop space quark fermion determinant eq.(1) is entirely supported at those loops C_{xx} with vanishing small area $S[C_{xx}]$ for large values of the noise-field vacuum streinght $g_\infty^2 \to +\infty$, since those of large area $S[C_{xx}]$ are suppressed on the loop space expression generating functional eq.(1) above mentioned, as much as similar K. Wilson mechanism for charge confining in Q.C.D.

Note that the same matter loop C_{xx} appearing in eq.(12) enters in the definition of all loop space objects in eq.(5). As a consequence, we have produced a loop space analysis supporting that at very large noise strenght ($g^{(\infty)} \to +\infty$), one has exactly the strong triviality of the $SU(N_c)$ on the sector of the quark abelian currents, in the mathematical sense that the dominant loops on the loop path integral eq.(5) are degenerate to the loop base point x or to the straight line vector bilinear quark field excitations trajectories motion. It yield as a result, thus

$$\lim_{M\to\infty} E_F^{(M)}(Z[J_\mu(x), B_\mu^{(M)}]) = \exp(0) = 1. \tag{14}$$

This result leads us to the conclusion that the theory has on free field behavior ([5]) at very strong noise-reservoir of the type introduced in this work signaling a kind of quantum field phenomena in a flavored dissapative vacuum media that destroys quantum phase coherence and leading to the theory's triviality as much as similar mechanism underlying the phenomena which has been obtained in ref. [1] for white-noise abelian reservoirs.

Acknowledgments:

The author thanks to the Mathematical Institute of UFF/Niterói, RJ for support.

References

[1] Luiz C.L. Botelho, Phys. Review D70, 045010 (2004);
Modern Physics Letters A, vol. 20, No 12, 929–946, (2005);
Eur. Phys. J.C 44, 267–276, (2005);

[2] G. Broda, Phys. Ref. Lett 63, 2709 (1989);
L.C.L. Botelho, J. Phys. A23, 1661, (1990);
Int. J. Mod. Phys. A 15, 755, (2000);
Mod. Phys. Letters 17B, No. 23, 1237, (2003);
J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Clarendon, Oxford, 2001).

[3] A.A. Migdal, Phys. Rep. 102, 199, (1983);
Luiz C.L. Botelho, J. Math. Phys. 30, 2160 (1989);
A.M. Polyakov, Gauge Field and Strings (Harwood Academic, Chur, Switzerland), 1987.

[4] A. Di Giacomo, H.G. Dosch, V.I. Shevchenko and Yu Simonov, Phys. Rep. 372, 319, (2002);

Luiz C.L. Botelho, Phys. Rev. 52D, 1729, (1995).

[5] E. Witten, Nucl. Phys. B149, 285, (1979);

B.L. Alteshuler and L.B. Joffe, Phys. Rev. Lett 69, 2979, (1992).

APPENDIX A

The Confining Property of the $U(\infty)$ - Charge Reservoir

We intend to show the own quantum decoherence/triviality of the $U(\infty)$-charged reservoir considered in the bulk of this work. Let us, thus, consider our translation invariant $U(M)$ non-abelian gauge field theory of the previous analysis. However defined in a finite volume domain Ω with $\text{vol}(\Omega) = ma^4$, where m is an positive integer with a playing the rule of a fundamental lenght scale associated to the elementary cell of volume a^4 of our finite-volume space-times (euclidean). We introduce at this point of our argument a closed loop C contained in the plane (μ,ν) – section of the domain $\Omega \subset R^4$ and possesing area (planar) $S[C_{\mu\nu}] = na^2$. (See eq.(7)).

$$\langle W[C]\rangle^{(\infty)} = \lim_{M\to\infty}\left(\frac{I_M[C]}{I_M[0]}\right) \tag{A-1}$$

The explicitly expressions for the objects on eq.(A-1) are the following C. Bollini and J.J. Giambiagi translation invariant gauge field path integrals ([5])

$$I_M[C] = \int_{-\infty}^{+\infty}\left(\prod_{a=1}^{M^2-M}\prod_{\mu=1}^{4} dA_\mu^a\right)\times\exp\left\{\frac{g^2}{2}(ma^4)\text{Tr}_{U(M)}([A_\mu,A_\nu]^2\right\}$$
$$\times\left(\frac{1}{M}\text{Tr}_{U(M)}P\left[\exp\left(ig\int_C A_\mu dx_\mu\right)\right]\right) \tag{A-2}$$

and

$$I_M(0) = \int_{-\infty}^{+\infty}\left(\prod_{a=1}^{M^2-M}\prod_{\mu=1}^{4} dA_\mu^a\right)\times\exp\left\{\frac{g^2}{2}(ma^4)\text{Tr}_{U(M)}([A_\mu,A_\mu]^2)\right\} \tag{A-3}$$

The exactly evaluation of eq.(A-2) was presented in our previous analysis, with the result below, after considering the re-scaling integration variable

$$A_\mu^a \to A_\mu^a\left[\frac{g^2}{2}(ma^4)+\frac{g^4(na^2)^2}{M}\right]^{-\frac{1}{4}} \qquad \mu = 1,2 \tag{A-4}$$

$$A_\mu^a \to A_\mu^a\left[\frac{g^2}{2}(ma^4)\right]^{-\frac{1}{4}} \qquad \mu \neq 1,2 \tag{A-5}$$

with the result

$$I_M[C] = \left[\frac{g^2}{2}(ma^4)+\frac{g^4(na^2)^2}{M}\right]^{-(M^2)}. \tag{A-6}$$

The same procedure is applied too in eq.(A-3) with the associated re-scaling $A_\mu^a \to A_\mu^a \left[\frac{g^2}{2}(ma^4)\right]^{-\frac{1}{4}}$. It yields the exactly result for the path-integral normalization factor

$$I_M[0] = \left[\frac{g^2}{2}(ma^4)\right]^{-M^2} . \qquad \text{(A-7)}$$

As consequence, we get the following result for the $U(\infty)$-Loop Wilson average ($(g^\infty)^2 = \lim_{M\to\infty}(g^2 M) < \infty$)

$$\begin{aligned}\langle W[C]\rangle^{(\infty)} &= \lim_{M\to\infty}\left(\frac{I_M[C]}{I_M[0]}\right) \\ &= \exp\left\{-(g_\infty)^2 \cdot \left(\frac{[na^2]^2}{ma^4}\right)\right\} \\ &= \exp\left\{-\left(\frac{(g_\infty)^2}{a^2}\right)\cdot\left[\left(\frac{n^2}{m}\cdot a^2\right)\right]\right\}. \end{aligned} \qquad \text{(A-8)}$$

At this point of our study we call the reader attention that in the final result eq.(A-8), we have considered already the case $D = 4$, where one must taken into account the transmutation phenomena of the Gauge coupling constant $g^{(\infty)}$ by considering the existence of a vacuum area domain a^2 (the cell of our space-time) as much as the famous "Q.C.D. spaghetti vacuum" of Nielsen, Olesen at. al. ([1]).

The area behavior of eq.(A-6) is easily obtained for large area loops $n^2 >> m$ in the following situation: If one considers the relationship $n = \gamma m$, with γ an adimensional number ($\gamma < 1$) which will be kept constant at the limit of infinite volume $m \to \infty$, one can see that eq.(A-8) gives area behavior for the Q.C.D. Wilson Loop for very large loop area

$$\begin{aligned}\langle W[C]\rangle^{(\infty)} &\underset{m\to\infty}{\sim} \exp\left\{-\left(\frac{\gamma(g_\infty)^2}{a^2}\right)\cdot na^2\right\} \\ &= \exp\left\{-\frac{(g_\infty)^2}{a^2_{\text{eff}}}\cdot \text{ Area } S[C]\right\}\end{aligned} \qquad \text{(A-9)}$$

At this point, one should envisage to implement a formal Feynman diagrammatic field theoretic $\frac{1}{M}$ – expansion on the finite order group $U(M)$ – Gauge theory by considering next translation – dependent field corrections on our reservoir field configurations of the form $A_\mu^a(x) = A_\mu^{(\infty)} + \frac{1}{M}G_\mu^a(x)$ in the usual Path-Integral measure with the matter confining behavior eq.(A-9) already built in the formalism, namely:

$$\prod_{\mu=1}^{D}\prod_{a=1}^{M^2-M} dA_\mu^a(x) = \prod_{\mu=1}^{D}\prod_{a=1}^{M^2-M} dA_\mu^a \cdot dG_\mu^a(x) \qquad \text{(A-10)}$$

$$
\begin{aligned}
&\exp\left\{-\frac{1}{2}\int_{\Omega}\mathrm{Tr}_{SU(M)}(F_{\mu\nu})^2(x)d^Dx\right\} \\
&\quad = \exp\Big\{-\frac{1}{2}\int_{\Omega}\mathrm{Tr}_{SU(M)}\Big((\partial_\mu G_\nu-\partial_\nu G_\mu)(x) \\
&\qquad + \frac{ig^{(\infty)}}{\sqrt{M}a}\Big[A_\mu+\frac{1}{M}G_\mu(x),A_\nu+\frac{1}{M}G_\nu(x)\Big]\Big)^2\Big\}. \qquad \text{(A-11)}
\end{aligned}
$$

It is worth remarking that the Feynman's Diagrammatic associated to the Back-Ground field decomposition in eqs.(A-10)–(A.11) leads to an exchange of "massive" Gluons and leading, thus, to a infrared-free perturbation analysis of the theory's observables.

APPENDIX B

On the law of large number in statistics

Let us present the usual mathematical methods procedure to define the large N llimit in Statistics.

The large N problem in Statistics starts by considering a set of N-independent random variables $\{X_\ell(w)\}_{\ell=1,\dots,N}$, with w belonging to a given fixed probability space $(\Omega, d\mu(w))$, besides of satisfying the following additionals constraints:

a) Theirs mean value posseses all the same value m:

$$\int_\Omega X_\ell(w)d\mu(w) = E\{X_\ell(w)\} = m \tag{B-1}$$

b) Theirs associated variance are all equals:

$$\sigma^2 \left[\left(\int_\Omega X_\ell^2(w)d\mu(w) \right)^2 - \left(\int_\Omega X_\ell(w)d\mu(w) \right)^2 \right] \tag{B-2}$$

The large N problem in Statistics can be stated now as the problem of defining mathematically the normalized limit of "large numbers" $N \to \infty$, of the sequence of random variables sum below

$$\lim_{N\to\infty} \hat{S}_N(w) = \lim_{N\to\infty} \left(\frac{1}{\sigma\sqrt{N}} \left(\sum_{\ell=1}^{N} X_\ell(w) - m \right) \right). \tag{B-3}$$

The path-integral solution for this problem contains all needed ideas and expose clearly the method which were implemented in our analysis in Gauge Field Theory.

Firstly, we define the associated Generating Functionals for each independent random variable $X_\ell(w)$, with $J \in R$. Namely:

$$\begin{aligned} Z_{\{X_\ell\}}((J)) = E\{e^{iJX_\ell(w)}\} &= \int_\Omega e^{iJX_\ell(w)} d\mu(w) \\ &= \sum_{k=0}^{\infty} \frac{i^k J^k}{k^1} \left(\int_\Omega (X_\ell(w))^k d\mu(w) \right) \end{aligned} \tag{B-4}$$

It is straightforward to see that the Generating Functional associated to the finite N

random variable sum eq.(3)

$$Z_N(J) = \prod_{\ell=1}^{N} \left[Z_{\{X_\ell\}} \left(J \left(\frac{X_\ell - m}{\sigma\sqrt{N}} \right) \right) \right]$$
$$= \left[\sum_{k=0}^{\infty} \frac{1}{k!} \frac{i^k M_k}{\sigma^k N^{k/2}} \cdot J^k \right]^N$$
$$= \left(1 - \frac{J^2}{2N} - \frac{iM_3 J^3}{6\sigma^3 N^{3/2}} + \frac{M_4 J^4}{24\sigma^4 N^2} + \dots \right)^N , \tag{B-5}$$

with the k-power averages given by the integral expressions below, which are supposed to be ℓ-independent

$$M_k = \int_\Omega (X_\ell(w)^k d\mu(w). \tag{B-6}$$

At this point, we define mathematically the large N limit by defining the effective statistics distribution parameters:

$$\lim_{N\to\infty} (\sigma\sqrt{N}) = \overline{\sigma}_{eff} < \infty \tag{B-7}$$

$$\lim_{N\to\infty} (mN) = \overline{m}_{eff} < \infty \tag{B-8}$$

and by taking the $N \to \infty$ limit of eq.(5) in the context of the definitions eq.(7)–eq.(8), by considering just for simplicity of our formulae writing $m = 0$ (see eq.(1)).

As a result, we have the simple expression below

$$\lim_{N\to\infty} [lg Z_N(J)] = N\, lg \left[1 - \frac{J^2}{2N} - \frac{iM_3 J^3}{6\sigma^2 N^{3/2}} + \frac{M_4 J^4}{24\sigma^4 N^2} + \dots \right]$$
$$= -\left(\frac{J^2}{2N} \right) N = -\frac{J^2}{2}, \tag{B-9}$$

or equivalently

$$\lim_{N\to\infty} Z_N(J) \equiv Z^{eff}_{N=\infty}(J) = e^{-\frac{J^2}{2}}, \tag{B-10}$$

which is nothing more than the Generating Functional associated to the Gaussian Statistics distribution:

$$Z^{eff}_{N=\infty}(J) = \frac{1}{\sqrt{2\pi} \cdot \overline{\sigma}} \times \int_{-\infty}^{+\infty} dx\, e^{-ixJ}\, e^{-\frac{x^2}{2\overline{\sigma}^2}} \tag{B-11}$$

which is formally the limit (with $m \neq 0$)

$$\lim_{N\to\infty} \left\{ \frac{1}{\sqrt{2\pi}(\sigma\sqrt{N})}\, e^{-\frac{(x-Nm)^2}{2N\sigma^2}} \right\} = \frac{1}{\sqrt{2\pi}\overline{\sigma}}\, e^{-\frac{(x-\overline{m})^2}{2\overline{\sigma}^2}} \tag{B-12}$$