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A Comment about the existence of a weak solution for a nonlinear Wave Damped Propagation

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Abstract

We give a proof for the existence of weak solutions on the initial-value problem of a non-linear wave damped propagation.

1 Introduction

One of the most important problems in the mathematical physics of the non-linear diffusion and wave-damped propagation is to establish the existence and the uniqueness of weak solutions in some convenient Banach spaces for the associated nonlinear evolution equation (see Refs 1–5). Another important class is these coming from initial-value problems for non-linear diffusion or damped hyperbolic partial differential equations with random initial conditions associated to the Gaussian processes sampled in certain Hilbert Spaces ([2]) and simulating the statistical turbulence physical phenomena ([6]).

The purpose of this short note is to contribute to such mathematical physics studies by using functional spaces compacity argument in order to produce proofs for the existence of weak solutions for a class of non-linear wave damped propagation on a smooth domain with Dirichlet boundary conditions and initial values belonging to the space $L^2(\Omega)$.

2 Existence Solution for Hyperbolic Non-linear Damping

Let us introduce firstly a non-linearity associated to a real valued function $F(x)$ on R such that $F'(x)$ is strictly positive and an external forcing source $f(x,t)$ supposed to be on the space $L^\infty([0,T],L^2(\Omega))$.

Associated to them we consider the following non-linear damped Hyperbolic initial value problem on $\Omega \times [0,T]$ with imposed Dirichlet boundary conditions and a damping positive constant ν.

$$\frac{\partial^2 U(x,t)}{\partial t^2} + (AU)(x,t) = -\nu \frac{\partial U(x,t)}{\partial t} + \Delta\left(F\left(\frac{\partial U}{\partial t}(x,t)\right)\right) + f(x,t) \tag{1}$$

The $L^2(\Omega)$-initial conditions are given by

$$U(x,0) = g(x) \in L^2(\Omega) \tag{2-a}$$

$$U_t(x,0) = h(x) \in L^2(\Omega) \tag{2-b}$$

$$U(x,t)|_{\partial\Omega} = 0 \tag{2-c}$$

and now the non-homogenous term $f(x,t)$ is considered to be a function belonging to the functional space

$$L^2([0,T]\times\Omega) \cap L^\infty([0,T], L^2(\Omega)). \tag{3}$$

We have thus the following theorem of existence (without uniqueness)

Theorem 2. *There exists a solution $\overline{U}(x,t)$ on $L^\infty([0,T]\times L^2)\Omega))$ for eq.(1)–eq.(3) in the weak sense with a test functional space as given by $C_0^\infty([0,T], H^2(\Omega)\cap H_0^1(\Omega))$.*

In order to arrive at such theorem, let us consider the a priori estimate for eq.(1) with $\dot{U}^{(n)} \equiv \frac{\partial}{\partial t}(U^{(n)}(x,t))$. Namely:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\dot{U}^{(n)}||^2_{L^2(\Omega)} + \nu||\dot{U}^{(n)}||^2_{L^2(\Omega)} \\
&+ (A\dot{U}^{(n)}, U^{(n)})_{L^2(\Omega)} + ||(F'(\dot{U}^{(n)}))^{1/2}\nabla\dot{U}^{(n)}||^2_{L^2(\Omega)} \\
&\quad = (f, \dot{U}^{(n)})_{L^2(\Omega)}
\end{aligned} \tag{4}$$

or equivalently

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{||\dot{U}^{(n)}||^2_{L^2(\Omega)} + (AU^{(n)}, U^{(n)})_{L^2(\Omega)}\right\} \\
&+ \nu\left\{||\dot{U}^{(n)}||^2_{L^2(\Omega)} + (AU^{(n)}, U^{(n)})_{L^2(\Omega)}\right\} \\
&+ ||(F'(\dot{U}^{(n)}))^{1/2}\nabla\dot{U}^{(n)}||^2_{L^2(\Omega)} \le \nu(AU^{(n)}, U^{(n)}) \\
&+ \frac{1}{2}\left(p||f||^2_{L^2(\Omega)} + \frac{1}{p}||\dot{U}^{(n)}||^2_{L^2(\Omega)}\right)
\end{aligned} \tag{5}$$

If one chooses here the integer p such that $\frac{1}{2p} = \nu$, we have the simple bound

$$\frac{1}{2}\frac{d}{dt}\{||\dot{U}^{(n)}||^2_{L^2(\Omega)} + (AU^{(n)}, U^{(n)})_{L^2(\Omega)}\} \le \frac{1}{4\nu}||f||^2_{L^2(\Omega)}. \tag{6}$$

As a consequence of eq.(6), there is a constant M such that the uniform bounds holds true (*even if for the case $T=+\infty$ for the case of $f\in L^2([0,\infty), L^2(\Omega))$*).

$$\sup\ \mathrm{ess}_{0\le t\le T}||\dot{U}^{(n)}||^2_{L^2(\Omega)} \le M \tag{7}$$

$$\text{sup ess}_{0\le t\le T}||U^{(n)}||^2_{L^2(\Omega)} \le M. \tag{8}$$

As a consequence of the bounds eq.(7) – eq.(8), there are two functions $\overline{U}(x,t)$ and $\overline{P}(x,t)$ such that we have the weak star convergence on $L^\infty([0,T]\times L^2(\Omega))$

$$\text{weak-star} \quad \lim_{n\to\infty}(U^{(n)}(x,t)) = \overline{U}(x,t) \tag{9}$$

$$\text{weak-star} \quad \lim_{n\to\infty}(\frac{\partial}{\partial t}U^{(n)}(x,t)) = \overline{P}(x,t). \tag{10}$$

We have thus that the relationship below hold true for any test function $v(x,t)\in C_c^\infty([0,T]\times H^2(\Omega)\cap H_0^1(\Omega))$ obviously satisfying the relations $v(x,0)=v(x,T)=\Delta v(x,0)=\Delta v(x,T)=v_t(x,0)=v_t(x,T)=v_{tt}(x,0)=v_{tt}(x,T)\equiv 0$ as a consequence of applying the Aubin-Lion theorem

$$\begin{aligned}\int_0^T dt\Big[&\left(\overline{U},\frac{d^2v}{d^2t}\right)_{L^2(\Omega)}+(\overline{U},Av)_{L^2(\Omega)}\\&+\nu\left(\overline{U},-\frac{dv}{dt}\right)_{L^2(\Omega)}-(F(\overline{P}),\Delta v)_{L^2(\Omega)}\Big]\\=\int_0^T& dt(f,v)_{L^2(\Omega)}\end{aligned} \tag{11}$$

with the initial conditions

$$\overline{U}(x,0)=g(x)\in L^2(\Omega) \tag{12}$$

$$\overline{P}(x,0)=h(x)\in L^2(\Omega). \tag{13}$$

Let us show now that

$$\overline{U}(x,t)=\int_0^t ds\overline{P}(x,s). \tag{14}$$

Firstly, let us remark that integrating on the interval $0\le t\le T$ the relationship eq.(4), one obtains the following estimate

$$\begin{aligned}&\frac{1}{2}\left(||\dot{U}^n(t)||^2_{L^2(\Omega)}-||\dot{U}^n(0)||^2_{L^2(\Omega)}\right)\\&+\left(\nu\int_0^t ds||\dot{U}_n||^2_{L^2(\Omega)}(s)\right)+\frac{1}{2}(AU^{(n)}(t),U^{(n)}(t))_{L^2(\Omega)}\\&-\frac{1}{2}(AU^{(n)}(0),U^{(n)}(0))+\int_0^t ds||(F'(\dot{U}^{(n)}))^{1/2}\nabla\dot{U}^{(n)}||^2_{L^2(\Omega)}\\&\le\frac{p'}{2}\int_0^t ds||f||^2_{L^2(\Omega)}+\frac{1}{2p'}\int_0^t||\dot{U}_n(s)||^2 ds\end{aligned} \tag{15}$$

Since the operator A satisfies the Gärding-Poincaré inequality on $L^2(\Omega)$

$$(AU^{(n)},U^{(n)})_{L^2(\Omega)}(t)\ge\gamma(\Omega)||U^{(n)}||^2_{L^2(\Omega)}(t), \tag{16}$$

one can see straightforwardly from eq.(15) by choosing $2p' > \frac{1}{\nu}$ and the previous bounds eq.(7) that there is a positive constant B such that

$$\int_0^T ds ||\dot{U}^{(n)}(s)||^2_{L^2(\Omega)} \leq B = MT \tag{17}$$

which by its turn yields that $\frac{dU_n(t,x)}{dt}$ is weakly convergent to $\overline{P}(x,t)$ in $L^2([0,T], L^2(\Omega))$.

As a consequence of general theorems of Function convergence on space of integrable functions (Aubin-Lion theorem again ([1])), one has that $\overline{P}(x,t)$ is the time-derivative of the function $\overline{U}(x,t)$ almost everywhere on Ω, since it is expected that $U^{(n)}(x,t)$ should be a strongly convergent sequence to $\overline{U}(x,t)$ on the separable and reflexive Banach Space $L^\infty([0,T] \times L^2(\Omega))$ ([1]). (See Appendix A for mathematical details).

APPENDIX A

Le us show that the functions $\overline{U}(x,t) \in L^{\infty}([0,T], L^2(\Omega))$ given by eq.(8) and $\overline{P}(x,t) \in L^{\infty}([0,T], L^2(\Omega))$ – eq.(9) are *coincident as elements* of the above written functional space of $L^2(\Omega)$ – valued essential bounded functions on $(0,T)$.

To verify such result, let us call the reader attention that since $U_m(x,t)$ is wearly convergent to $U(x,t)$ in $L^{\infty}([0,T], L^2(\Omega))$, we have that the set $\{U_m(x,t)\}$ is convergent to the function $U(x,t)$ as a Schwartz distribution on $L^2(\Omega)$ since $D([0,T], L^2(\Omega)) \subset L^1([0,T], L^2(\Omega))$.

This means that

$$U_m(x,t) \to U(x,t) \quad \text{in} \quad D'([0,T], L^2(\Omega)). \tag{1-A}$$

Analogous result hold true for the time-derivative of the above written equation as a result of $U(x,t)$ be a function

$$\frac{\partial}{\partial t} U_m(x,t) \to \frac{\partial}{\partial t} U(x,t) \quad \text{in} \quad D'([0,T], L^2(\Omega)) \tag{2-A}$$

By the other side, the set $\left\{\frac{\partial U_m(x,t)}{\partial t}\right\}$ converges wearly star in $L^1([0,T], L^2(\Omega))$ to $\overline{P}(x,t) \in L^{\infty}([0,T], L^2(\Omega))$ which, by its turn, means that

$$\frac{\partial U_m(x,t)}{\partial t} \to \overline{P}(x,t) \quad \text{in} \quad D'([0,T], L^2(\Omega)). \tag{3-A}$$

By the uniqueness of the limit on the distributional space $D'([0,T], L^2(\Omega))$, we have the coincidence of $\frac{\partial U(x,t)}{\partial t}$ and $\overline{P}(x,t)$ as elements of $D'([0,T], L^2(\Omega))$. However, $\overline{P}(x,t)$ is a function, so by general theorems on Schwartz distribution theory $\left\{\frac{\partial U(x,t)}{\partial t}\right\}$ must be a function either since $L^2(\Omega)$ is a separable Hilbert Sapce. As a consequence we have that $\frac{\partial U(x,t)}{\partial t} = \overline{P}(x,t)$ as elements of $L^{\infty}([0,T], L^2(\Omega))$, which is the result searched

$$\frac{\partial U(x,t)}{\partial t} = \overline{P}(x,t) \quad \text{a.e. in} \quad ([0,T] \times \Omega. \tag{4-C}$$

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